Ordinal Models

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Today: Ordinal Models

Topics of Course

- Proportional Odds Model.

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- Alternatives to Proportional Odds.

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- Alternatives to Proportional Odds.
- Multinomial Models.

Basic Issues

- Binary Logit.

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- Derivation of Model: A Coin-Flip

$$(y_i \mid n_i, \mathbf{x}_i) \sim \text{Bin}(n_i, p_i \mid \mathbf{x_i})$$

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$$(y_i \mid n_i, \mathbf{x}_i) \sim \text{Bin}(n_i, p_i \mid \mathbf{x_i})$$

- Log-Likelihood for the sample, given binomial distribution:

$$\sum_{i=1}^{N} \left[\log \binom{n_i}{y_i} + y_i \log p_i + (n_i - y_i) \log(1 - p_i) \right]$$

Rearranging, gives

$$\sum_{i=1}^{N} \left[y_i \log \left(\frac{p_i}{1 - p_i} \right) + n_i \log(1 - p) + \log \binom{n_i}{y_i} \right]$$

- This should look familiar: it's a the log-likelihood for a "coin flip."

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- The Canonical Link function equates ("links") the linear model to the canonical parameter (see Simonoff, p. 367):

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 x_{1i} + \ldots + \beta_k x_{ki}.$$

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- Z is the linear predictor.

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- Basic problems emerge but are often ignored.
- Regression is commonly applied:
  "Because the dependent variables are categorical, OLS regression is technically inappropriate. We found substantially the same results, however, using ordinal logit models. We report the OLS results because their interpretation is more straightforward." (Zuckerman and Jost, 2001 SPQ).

Illustration

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- `R` Code:

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- Alternatives?

Basic Issues

- Ordinal scales are prevalent in survey research
- Frequently used as response variables
- Interest in $Y = f(\mathbf{x}'\beta)$
- Normal-theory methods commonly applied (OLS)
- Cumulative link models also applied
- Either strategy often works poorly

Common Issues That Arise

- Equal-interval scoring usually unrealistic
- "Parallel" regression assumption frequently will not hold
- Ordinality may not exist, conditional on covariates **x**
- Alternative contrasts may be of interest
- Several alternative models are considered

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$$\begin{aligned} Y = 1 &\equiv Y^* \leq \alpha_1 \\ Y = 2 &\equiv \alpha_1 < Y^* \leq \alpha_2 \\ Y = 3 &\equiv \alpha_2 < Y^* \leq \alpha_3 \\ Y = 4 &\equiv Y^* > \alpha_3 \end{aligned}$$

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- Specify C.D.F. for ϵ (logistic, standard normal, cloglog)
- With logistic, proportional odds is obtained.

Proportional Odds Model

- Gives rise to:

$$\Pr(Y \leq y_j \mid \mathbf{x}) = \frac{\exp(\alpha_j - \mathbf{x}'\beta)}{1 + \exp(\alpha_j - \mathbf{x}'\beta)}$$

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- Linear Model for log-odds:

$$\log\left[\frac{\Pr(Y \leq y_j \mid \mathbf{x})}{\Pr(Y > y_j \mid \mathbf{x})}\right] = \alpha_j - \mathbf{x}'\beta, \qquad j = 1, 2, \ldots j-1$$

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- Proportional Odds Property:

$$\frac{\exp(x_1\beta)}{\exp(x_2\beta)} = \exp\{(x_1 - x_2)'\beta\}$$

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- Application

Application

Table 1: Support for Affirmative Action
OLS and Proportional Odds Estimates

Variable	Estimate	
	OLS	Proportional Odds
Symbolic Racism	1.375	2.562
	(0.111)	(0.207)
Racial Prejudice	0.383	0.728
	(0.193)	(0.356)
Education	0.259	0.510
	(0.102)	(0.182)
Ideology	0.066	0.117
	(0.027)	(0.047)
Female	0.025	0.075
	(0.049)	(0.088)
Intercept	1.479	
	(0.127)	
α_1		0.652
		(0.233)
α_2		1.888
		(0.236)
α_3		3.268
		(0.245)
n	1744	1744
log-likelihood:		−2289.518

Proportionality Tests

Table 2: Testing the Proportional Odds Assumption

Variable	Coefficient		
	$y > 1$	$y > 2$	$y > 3$
Symbolic Racism	1.822	2.458	3.037
Racial Prejudice	1.384	0.219	0.664
Education	0.209	0.496	0.651
Ideology	0.116	0.105	0.139
Female	-0.118	0.144	0.052
Constant	-0.279	-1.651	-3.645

Variable	χ^2	$p > \chi^2$	df
All	29.33	0.001	10
Symbolic Racism	12.37	0.002	2
Racial Prejudice	8.15	0.017	2
Education	2.29	0.319	2
Ideology	0.33	0.849	2
Female	5.79	0.055	2

Nonproportional Odds

- Generalized Model:
$$\log\left[\frac{\Pr(Y \leq y_j|\mathbf{x})}{\Pr(Y > y_j|\mathbf{x})}\right] = \alpha_j - \mathbf{x}'\beta_\mathbf{j}, \qquad j = 1, 2, \ldots j-1$$

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- Partial Proportional Odds:
$$\log\left[\frac{\Pr(Y\leq y_j|\mathbf{x})}{\Pr(Y>y_j|\mathbf{x})}\right] = -\alpha_j - \mathbf{x}'\beta - \mathbf{t}'\gamma_\mathbf{j}, \qquad j = 1, 2, \ldots j-1$$

Nonproportional Odds

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$$\log\left[\frac{\Pr(Y\leq y_j|\mathbf{x})}{\Pr(Y>y_j|\mathbf{x})}\right] = \alpha_j - \mathbf{x}'\beta_\mathbf{j}, \quad j = 1, 2, \ldots j-1$$
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- Restricted Generalized Logit:
$$\log\left[\frac{\Pr(Y\leq y_j|\mathbf{x})}{\Pr(Y>y_j|\mathbf{x})}\right] = \alpha_j + \mathbf{x}'\beta + \mathbf{z}'\zeta_\mathbf{j}, \quad j = 1, 2, \ldots j-1$$

Application: UPP

Table 3: Support for Affirmative Action
Partial Proportional Odds (UPP Model)

Variable	Coefficient	Standard Error
Symbolic Racism	1.788	(0.285)
γ_2	0.800	(0.257)
γ_3	1.345	(0.358)
Racial Prejudice	1.563	(0.495)
γ_2	-1.437	(0.440)
γ_3	-0.811	(0.586)
Female	-0.098	(0.125)
γ_2	0.277	(0.111)
γ_3	0.147	(0.145)
Education	0.479	(0.182)
Ideology	0.122	(0.047)
α_1	.273	(.22)
α_2	-1.635	(.203)
α_3	-3.263	(.236)
n	1744	
log-likelihood	-2274.293	

Application: Restricted Generalized Ordinal

Table 3a: Support for Affirmative Action
Partial Proportional Odds (Generalized Ordinal Logit)

Variable	Coefficient		
	C_1	C_2	C_3
Symbolic Racism	1.788	2.588	3.133
	(0.285)	(0.245)	(0.280)
Racial Prejudice	1.563	0.127	0.752
	(0.495)	(0.421)	(0.462)
Education	0.478	0.478	0.478
	(0.182)	(0.182)	(0.182)
Ideology	0.122	0.122	0.122
	(0.047)	(0.047)	(0.047)
Female	-0.098	0.180	0.049
	(0.125)	(0.101)	(0.113)
Constant	-0.509	-1.698	-3.639
	(0.283)	(0.261)	(0.299)
n	1744		
log-likelihood	−2274.293		